Freely floating cylinder on a 3D fluid governed by the Boussinesq equations

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Abstract

We study a model describing the interaction between waves at the surface of a sea and a freely floating partially immersed cylinder in the Boussinesq regime. This wave-interaction system can be reduced to an initial boundary value problem for the Boussinesq equations in an exterior domain. The boundary condition depends on the motion of the floating cylinder which is in turn determined by a nonlinear ODE with forcing terms coming from the exterior wave field.

Keywords: Fluid-structure interaction, Dispersive perturbation of hyperbolic boundary value problems, Finite volume.

1 Introduction

The mathematical study of floating objects at the surface of water is a key issue for collecting water-waves energy. One considers the case of a rigid cylinder of radius $R > 0$ with vertical walls and a flat bottom. Moreover, we will assume that it can only vertically move (see figure 1).

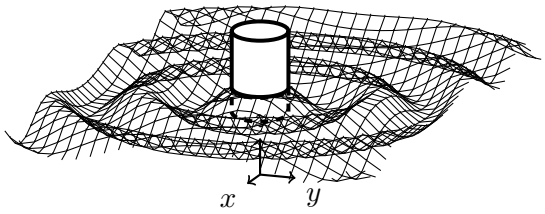


Figure 1: Waves propagation with a floating cylinder in the case where the flow is initially axisymmetric without swirl.

The propagation domain is divided into two areas:

- $B(0,R)$ where the floating cylinder is located,
- $\mathcal{E} := \mathbb{R}^2/B(0,R)$ where the waves move freely.

The contact line between the two areas will be denoted by $\Gamma := \overline{B(0,R)} \cap \overline{\mathcal{E}}$. We will denote by e_r the outward normal of Γ and

$$[f]_\Gamma := \lim_{h\to 0}[f(R+h) - f(R-h)],$$

denotes the jump at the contact line.

2 Wave-structure model

The motion of the floating cylinder is governed by the Newton equation

$$m\ddot{\delta} + mg = \int_{B(0,R)} \frac{P}{\varepsilon}, \tag{1}$$

where δ is the distance of the center of mass to its equilibrium and $P : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is the pressure at the the surface of the sea. The motion of the waves is governed by the $2D$ Boussinesq-Abbott system

$$\begin{cases} \partial_t \zeta + \operatorname{div} Q = 0, \\ T_\kappa \partial_t Q + \varepsilon \operatorname{div}\left(\frac{Q\otimes Q}{h}\right) + h\nabla(\zeta + \frac{P}{\varepsilon}) = 0, \\ h := 1 + \varepsilon\zeta\,, \quad T_\kappa := (1 - \kappa^2\nabla\operatorname{div}), \end{cases} \tag{2}$$

where $\zeta : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is the elevation of waves, $h : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is the height of the water column and $Q : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is the discharge. Note that $\varepsilon > 0$ and $\kappa > 0$ are dimensionless parameters.

The pressure P is unknown under the floating cylinder, the interior domain $B(0,R)$, and constant equals to the atmospheric pressure in the exterior domain:

$$P_{|_\mathcal{E}}(t,x) = P_{\text{atm}}.$$

The wave elevation ζ is free in the exterior domain $\mathcal{E}$ but be constraint in the interior domain

$$\zeta_{|_{B(0,R)}}(t,x) = \delta(t) + \zeta_{\text{eq}}\,,\ \zeta_{\text{eq}} = \varepsilon^{-1}(h_{\text{eq}} - 1),$$

with h_{eq} the height of the water column at equilibrium in $B(0,R)$. The pressure P could be understand as the Lagrange multiplier associated to the constraint $\zeta_{|_{B(0,R)}} = \delta + \zeta_{\text{eq}}$. Finally to complete the system, we imposed transmission conditions at the contact line Γ:

1. (conservation of the fluid volume)

$$[Q \cdot e_r]_\Gamma = 0,$$

2. (conservation of the energy of the whole fluid-structure system)

$$\left[\frac{P}{\varepsilon} + \zeta + \kappa^2 \frac{\ddot{\zeta}}{h} + \frac{|Q|^2}{2h^2}\right]_\Gamma = 0,$$

3. (no vortex line on the contact line)

$$\left[\frac{Q}{h} \times e_r\right]_\Gamma = 0.$$

3 Return to equilibrium

The return to equilibrium corresponds to the situation where the fluid is initially at rest whereas the solide is dropped from a position out of equilibrium. In that case, the flow is axisymmetric without swirl, *i.-e.* $\zeta = \zeta(t,r)$ and $Q = q(t,r)e_r$, where r stands for the radial coordinate. In [3] such a configuration was considered for the hyperbolic nonlinear shallow water equations (equation (2) with $\kappa = 0$). The Boussinesq equations (2) are a dispersive perturbation of the hyperbolic nonlinear shallow water equations. This dispersive perturbation induces drastic changes on the behavior and analysis of the associated initial boundary value problem. In that situation, the wave-interaction system can be reduced to an initial boundary value problem for the Boussinesq equations (2) in the exterior domain $\mathcal{E}$ with the following boundary condition

$$Q_{|\mathcal{E}} = -\frac{R}{2}\dot{\delta} e_r \quad \text{on} \quad \Gamma, \tag{3}$$

which is in turn determined by a nonlinear ODE with forcing terms coming from the exterior wave field. To give this ODE, one first introduce, the dispersif operators

$$\mathfrak{R}_\alpha f = u, \quad \forall \alpha \in \{0,1\},$$

such that for any $f \in L^2(rdr)$, u is the $H^2(rdr)$-solution of

$$\begin{cases} \left[1 - \kappa^2\big(\partial_r + \alpha\frac{\cdot}{r}\big)\big(\partial_r + (1-\alpha)\frac{\cdot}{r}\big)\right]u = f, \quad \forall r > R, \\ (1-\alpha)u_{|_{r=R}} + \alpha\partial_r u_{|_{r=R}} = 0. \end{cases}$$

One can show using the Newton equation (1) and the dispersion in (2) that the time-depend variable

$$\mathfrak{z} := (\delta, \underline{\zeta}) \text{ where } \underline{\zeta} := \lim_{h\to 0} \zeta_{|\mathcal{E}}(t, R+h),$$

is solution of the following ODE

$$M(\mathfrak{z})\ddot{\mathfrak{z}} + \mathcal{Z}(\mathfrak{z}, \dot{\mathfrak{z}}) = \begin{pmatrix} 0 \\ \mathfrak{f}_{\text{hyd}} \end{pmatrix} \tag{4}$$

where

$$M(\mathfrak{z}) := \begin{pmatrix} m + \frac{\pi R^4}{8(h_{\text{eq}}+\varepsilon\delta)} + \kappa^2 \frac{\pi R^3}{h_{\text{eq}}+\varepsilon\delta} & \kappa^2 \frac{\pi R^2}{1+\varepsilon\underline{\zeta}} \\ \kappa^2 & -\frac{\kappa}{2}\Big(\frac{RK_1'(\frac{R}{\kappa})}{K_1(\frac{R}{\kappa})} + \kappa\Big) \end{pmatrix},$$

is a positive-definite matrix (K_1 stands for the modified Bessel of the second kind), $\mathcal{Z}$ is a Lipschitz-continuous field and the hydrodynamical force is

$$\mathfrak{f}_{\text{hyd}} := \left[\mathfrak{R}_1\left(\zeta + \frac{\zeta^2}{2} + \varepsilon\frac{q^2}{h}\right) + \kappa^2\big(\partial_r + \frac{\cdot}{r}\big)\mathfrak{R}_0\big(\frac{\varepsilon q^2}{hr}\big)\right]_{r=R}.$$

Thus, we have written the wave-structure equations in a such way that there is no need to compute the pressure exerted at the bottom of the object which circumvents the main numerical difficulty. Based on this formulation and in the spirit of [2], we propose a first order scheme that involves a generalization of the HLL scheme with nonlocal flux and a source term to solve (2,3) in $\mathcal{E}$, which is coupled to a first order Euler scheme for the ODE (4). Note that (4) gives the trace of $Q_{|\mathcal{E}}$ and $\zeta_{|\mathcal{E}}$ on the boundary Γ that is very helpful.

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References

[1] G. Beck, D. Lannes. Freely Floating Objects on a Fluid Governed by the Boussinesq Equations. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 2022, 39 (3),

[2] G. Beck, D. Lannes, L. Weynans, A numerical method for wave-structure interactions in The Boussinesq regime, 2023. Preprint hal-04151128

[3] E. Bocchi, Floating structures in shallow water: local well-posedness in the axisymmetric case, *SIAM Journal on Mathematical Analysis* 52 (2020), 306-339.

[4] D. Lannes, T. Iguchi, The $2D$ nonlinear shallow water equations with a partially immersed obstacle, 2023. Preprint