The return to the equilibrium of a floating cylinder in the Boussinesq regime.

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Abstract

In this work we study the interactions between waves at the surface of a sea and a partially immersed moving floating cylinder in the Boussinesq regime. The modelisation of the full waves-structure system can be written as a system of PDEs combined to a system of ODEs. Here we have an interest in the case of the return to the equilibrium in the linear regime, that is to say the solid is dropped in a fluid at rest without any speed. In this case we show that the solid returns to an equilibrium position in infinite time, but slower than $\mathcal{O}(t^{-1/2})$. Also we intend to give two numerical schemes for this system.

Keywords: Fluid-structure interactions, Discrete stability estimates

1 Modelisation

We study the case of a freely floating cylinder of radius $R > 0$ and of mass m in a fluid. The cylinder is supposed to be allowed to move only vertically. We consider that there is no swirl in the fluid or at its surface, and that there is an axisymmetry with respect to the central axis of the cylinder.

With the previous considerations, the 2D situation can be written as a 1D problem with radial terms and we reduce the study to a slice of the problem, that is to say on $[0, +\infty)$. The behavior of the waves is governed by the dimensionless Boussinesq-Abbott system:

$$\begin{cases} \partial_t \zeta + \mathrm{d_r}\, q = 0, \\ D_\kappa \partial_t q + \partial_r \zeta = 0, \\ \mathrm{d_r} = \partial_r + \frac{d-1}{r}, \quad D_\kappa := (1 - \kappa^2 \partial_r \mathrm{d_r}) \end{cases} \tag{1}$$

where $d \in \{1, 2\}$ is the dimension of the problem. In fact we can treat both the 1D and the 2D situations simultaneously (see [3] for the 1D problem, and [1] for the 2D problem). The shallowness is denoted by κ. Mathematically κ describes the dispersion. The variables are ζ, for the elevation of the waves and q for the discharge.

It was shown in [1, 3] that (1) can only be studied in the domain $r > R$ where the wave's elevation is free, with boundary condition $\underline{q} = -\frac{R}{2}\dot{\delta}$, where δ is the distance of the center of mass to its equilibrium and $\underline{q} = \lim_{\eta \to 0^+} q(R + \eta)$. Also δ is described by the Newton's equation.

2 Some theoritical results

Here we study the case of the return to the equilibrium in the linear case. That is to say at the initial time $t = 0$ the solid is dropped without any speed in a fluid at rest. All the following results are showed in [3] for $d = 1$ and in [1] for $d = 2$.

By inversing the dispersive operator D_κ we obtain the *hidden equation*:

$$\kappa^2 \ddot{\zeta} + \zeta = \mathcal{R}\zeta + B(r/\kappa)(\underline{\mathcal{R}\zeta} - \delta), \quad r \geq R. \tag{2}$$

where B is given by decreasing exponential (resp. Bessel functions) if $d = 1$ (resp. if $d = 2$). The operator $\mathcal{R}$ is the dispersive operator such that for any $u \in L^2((R, +\infty))$ we define $\mathcal{R}u \in H^2((R, +\infty))$ by solving the elliptic problem:

$$(1 - \kappa^2 \mathrm{d_r} \partial_r)\mathcal{R}u = u, \quad \underline{\partial_r \mathcal{R}u} = 0. \tag{3}$$

The Newton's equation contains the pressure force exerted by the fluid under the solid. We can obtain an expression of this pressure term depending on ζ. If we replace it in the Newton's equation and combine the result with the hidden equation it leads to:

$$M_{R,\kappa} \ddot{\delta} + \delta = \underline{\mathcal{R}\zeta} \tag{4}$$

where $M_{R,\kappa} = m + m_{add}$ where $m_{add} > 0$ is the *added mass*, representing the inertia of the solid.

We showed [1, 3] that the model is equivalent to the *augmented formula*, satisfied in the domain $r > R$. It is formulated with the *augmented variables* $u := (\zeta, q)$ for the fluid's variables and the boundary variable $Z := (\delta, \underline{\zeta})$:

$$\begin{cases} \partial_t u + \partial_r \mathcal{F}_\kappa[u] + \frac{1}{r} A[u] = \kappa(\underline{\mathcal{R}\zeta} - \delta)\gamma, \; r > R, \\ M\ddot{Z} + TZ = \mathfrak{F}(\underline{\mathcal{R}\zeta}), \\ \underline{u} = (\underline{\zeta}, -\frac{R}{2}\dot{\delta}), \quad (u, Z)_{|t=0} = (u_0, Z_0). \end{cases} \tag{5}$$

The terms $\mathcal{F}_\kappa$, A, γ, $\mathfrak{F}$, M and T are explicited in [1] and [3]. γ is a dispersive boundary layer exponentially decreasing. Since $\underline{\mathcal{R}\zeta}$ requires to know ζ on $(R, +\infty)$, the PDEs system and the ODEs system are coupled.

All the equations (2), (4) and (5) have an expression in the nonlinear regime. In particular in (5) $\mathcal{F}_\kappa$ is still a flux in the nonlinear-regime. With the augmented formula we show that the linear system is globally wellposed and that the nonlinear system is locally wellposed.

In the return to the equilibrium the interesting quantity is δ and one wants to avoid computing the other variables. We showed in [1] that δ satisfies a Cummin's type equation both in 1D and 2D. It provides an expression of the Laplace transform of δ depending only on $\delta_{|_{t=0}}$. Sharp estimates on this Laplace transform allows to show that δ lies in $H^2(\mathbb{R}^+)$ and we showed that δ and $\dot{\delta}$ decay to 0 slower than $t^{-1/2-\rho}$ for any $\rho \in (0, 1/2)$, *i.e* for large t, there exists $c > 0$ such that $|\delta(t), \dot{\delta}(t)| > ct^{-1/2-\rho}$.

3 Numerical simulations

The difficult part of the numerical simulations is the boundary $r = R$: the initial problem (1) gives only one boundary condition on q, given by δ, coupled with the Newton's equation, but the classical finite volume method requires to know all the quantities at the boundary. We overpass this problem by using the hidden equation (2) in $r = R$. The theorical analysis provides several methods to make numerical simulations of the problem. A preliminary step consists in discretizing (3) as an elliptic problem.

3.1 The flux method.

The first method is based on (5) and is an adaptation of [4] in the 2D case. The simulations are done through a finite volume method, where the left boundary cell is filled with the order 2 ODE on Z, which gives $\underline{u}$. This scheme is explicit in time. In 1D, such a scheme needs adding an artificial viscosity to make the scheme stable. It is quite usual to do it, and in [4] the simulations were conclusive. Therefore in 2D we added viscosity with the same philosophy. The interesting point of this method is that it is adaptable to the non-linear regime because the nonlinearities just modify the flux. For this method, a computation of simultaneously ζ and q is necessary on the contrary to the second one.

This method does not fully use the hidden equation (2). Indeed we only use it at $r = R$ whereas it is satisfied on the entier domain $r \geq R$. It leads to the second method.

3.2 The ODE method.

This one uses the ODE formulation, which is the combination of (2) and (4). Indeed thanks to the regularizing properties of $\mathcal{R}$, (2) is an ODE on each space's point. Thanks to this formulation, the only importance of the space discretization is contained in the discretization of (3). Therefore the only discussion is on the time discretization. The coupling terms ($\underline{\mathcal{R}\zeta}$ and δ in the hidden equation) are explicited. However the nonlinearities involve a coupling between ζ and q. It might be difficult to adapt the scheme to the nonlinear regime.

This scheme opens several perspectives: we almost get rid of the space discretization, that impacts the scheme only through $\mathcal{R}\zeta$ and the CFL can be independent on the space step. Also we can get rid of the first cell in ζ and avoid to compute it. Furthermore higher order schemes are reachable with this method.

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