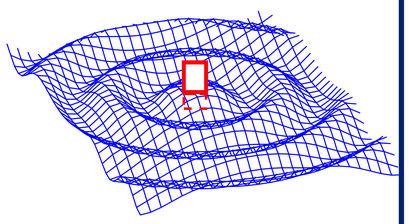
# Vertical motion of a freely floating cylinder in Boussinesq regime.

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## 3D situation

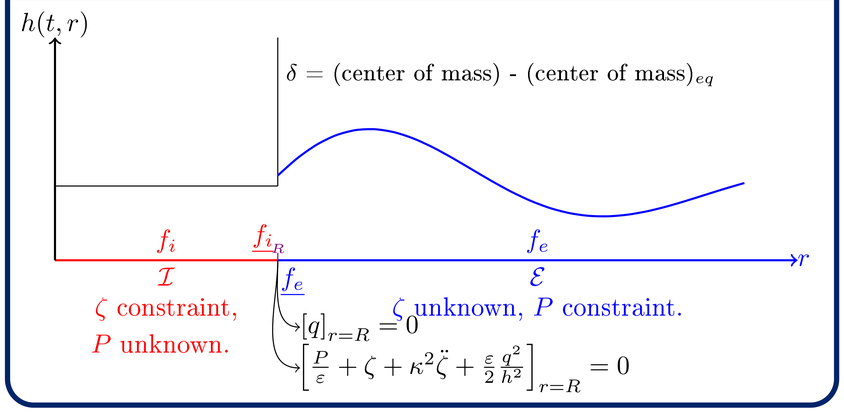


## Hypothesis

- Axisymetry without swirl,
  - Discharge: $q(t,r)\vec{e_r}$,
  - Wave's elevation: $\zeta(t,r)$,
- The fluid fit the bottom of the solid in $\mathcal{I}$.

$$d_r := \partial_r + \frac{\cdot}{r}$$

## 1D Radial situation




## Augmented formulation

$$\begin{cases} \partial_t\zeta + d_r q = 0, \\ \partial_t q + \partial_r\mathfrak{R}_1\left(\zeta + \varepsilon\frac{\zeta^2}{2} + \varepsilon\frac{q^2}{h}\right) + \varepsilon\mathfrak{R}_0\frac{q^2}{rh} = \partial_t\underline{q}G, \end{cases} \qquad r > R$$

$$\underline{q} = -\frac{R}{2}\dot{\underline{\delta}},$$

and

$$\begin{pmatrix}\text{added mass eq}\\ \text{hidden trace eq}\end{pmatrix} \quad \begin{pmatrix}\tau_\kappa^2(\varepsilon\delta)\ddot{\delta}\\ \kappa^2\ddot{\underline{\zeta}}\end{pmatrix} = \mathfrak{F}\left(\delta, \dot{\delta}, \zeta, \dot{\underline{\zeta}}, \underline{\mathfrak{R}_1 f}\right), \qquad (1)$$

with $\mathfrak{R}_\alpha$, $\alpha \in \{0,1\}$ dispersive nonlocal operators defined as

$$\begin{cases} \left[1 - \kappa^2\left(\partial_r + \alpha\frac{\cdot}{r}\right)\left(\partial_r + (1-\alpha)\frac{\cdot}{r}\right)\right]\mathfrak{R}_\alpha u = u, \quad \forall r > R, \\ (1-\alpha)\mathfrak{R}_\alpha u_{|_{r=R}} + \alpha\partial_r\mathfrak{R}_\alpha u_{|_{r=R}} = 0. \end{cases}$$

and $G$ dispersive boundary layer defined as $\begin{cases} (1-\kappa^2\partial_r d_r)G = 0, \\ G(r=R) = 1. \end{cases}$

## How to get the added mass equation

Combining (3) with (2) in $\mathcal{I}$ leads to obtain the added mass equation (1) with

$$\tau_\kappa^2(\varepsilon\delta) := \tau_{buoy}^2 + \kappa^2 + \frac{R^2}{8(1+\varepsilon\delta)} > \tau_{buoy}$$

## How to get the hidden equation

$$D_\kappa\partial_t q + \varepsilon d_r\frac{q^2}{h} + h\partial_r\zeta = 0$$

$$\Downarrow D_\kappa^{-1}$$

$$\partial_t q + \partial_r\mathfrak{R}_1\left(\zeta + \varepsilon\frac{\zeta^2}{2} + \varepsilon\frac{q^2}{h}\right) + \varepsilon\mathfrak{R}_0\frac{q^2}{rh} = \underline{\partial_t q}G$$

$$\Downarrow \kappa^2 d_r$$

$$\kappa^2\ddot{\zeta} + \zeta + \varepsilon\frac{\zeta^2}{2} + \varepsilon\frac{q^2}{h} = \ddot{\delta}\,\kappa^2\frac{R}{2}d_r G + \mathfrak{R}_1\left(\zeta + \varepsilon\frac{\zeta^2}{2} + \varepsilon\frac{q^2}{h}\right) + \varepsilon\kappa^2 d_r\mathfrak{R}_0\frac{q^2}{rh}, \quad r > R$$

## Modelisation

Let $\varepsilon \in [0,1)$ and $\kappa \in (0,1)$.

- Boussinesq-Abbott

$$\begin{cases} \partial_t\zeta + d_r q = 0, \\ D_\kappa\partial_t q + \varepsilon d_r\frac{q^2}{h} + h\partial_r\zeta = -h\partial_r\frac{P}{\varepsilon}, \\ h = 1 + \varepsilon\zeta, \\ D_\kappa = (1 - \kappa^2\partial_r d_r), \end{cases} \qquad (2)$$

- Newton's motion law

$$\tau_{buoy}^2\ddot{\delta} + \frac{1}{\varepsilon}m = \frac{1}{\mathcal{I}}\int_{\mathcal{I}}\frac{P}{\varepsilon}. \qquad (3)$$

## Local wellposedness

$$(\zeta_{in}, q_{in}) \in H^1(\mathcal{E}) \times H^2_\kappa(\mathcal{E}), \delta_{in} \in \mathbb{R},$$
$$-R\dot{\delta}_{in} = 2q_{in}(r=R), \underline{\zeta_{in}} = \zeta_{in}(r=R), \dot{\underline{\zeta}}_{in} = -\underline{d_r q_{in}}$$

$$\Downarrow$$

$$(\zeta,\ q,\ \delta,\ \dot{\delta},\ \underline{\zeta},\ \dot{\underline{\zeta}}) \in \mathscr{C}^\infty\left([0,T], H^1(\mathcal{E}) \times H^2_\kappa(\mathcal{E}) \times \mathbb{R}^4\right),$$

$$T = \mathscr{O}\left(min\left(\varepsilon^{-1}\kappa,\ \varepsilon^{-1/2}\right)\right)$$

with $H^2_\kappa(\mathcal{E})$ the space $H^2(\mathcal{E})$ equipped with the norm

$$||\cdot||_{H^2_\kappa(\mathcal{E})} = ||\cdot||_{H^1(\mathcal{E})} + ||\kappa\partial_r d_r\cdot||_{L^2(\mathcal{E})}$$

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Sketch of the proof: Fixed point argument based on the iterative scheme

$$\begin{cases} \partial_t\zeta^{k+1} + d_r q^{k+1} = 0, \\ \partial_t q^{k+1} + \partial_r\mathfrak{R}_1\left(\zeta^{k+1} + \varepsilon\frac{\zeta^{k\,2}}{2} + \varepsilon\frac{q^{k\,2}}{h^k}\right) + \varepsilon\mathfrak{R}_0\frac{q^{k\,2}}{rh^k} = \partial_t\underline{q}^{k+1}G, \\ + \\ \text{Same for the linearized version of (1)} \end{cases} \qquad r > R$$

## Numerical simulations




Suppose $u^n := (\zeta^n, q^n), \delta^n,\ \dot{\delta}^n,\ \underline{\zeta}^n,\ \dot{\underline{\zeta}}^n$ are known, $\ddot{\delta}^n$ is given by (1):

1. Solve the elliptic problem $\rightarrow \mathfrak{R}_1 u^n, \mathfrak{R}_1 f^n, \mathfrak{R}_0\frac{q^2}{rh}{}^n, G$, with $f := \zeta + \varepsilon\frac{\zeta^2}{2} + \varepsilon\frac{q^2}{2}$,
2. For $i > 1$, compute $u_i^{n+1}$ by

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right) - \Delta t\begin{pmatrix}\frac{q_i^n}{r_i}\\ \varepsilon\mathfrak{R}_0\frac{q^2}{rh}{}_i^n + \frac{R}{2}\ddot{\delta}^n G_i\end{pmatrix}$$

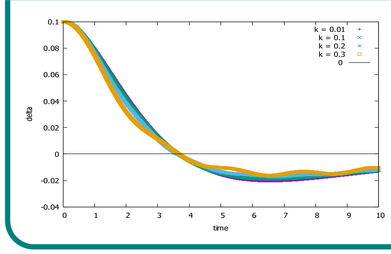
$$\mathcal{F}_{i+1/2} := \frac{1}{2}\begin{pmatrix}q_{i+1} + q_i\\ (\mathfrak{R}_1 f)_{i+1} + (\mathfrak{R}_1 f)_i\end{pmatrix} - \frac{\lambda}{2}\begin{pmatrix}(\mathfrak{R}_1\zeta)_{i+1} - (\mathfrak{R}_1\zeta)_i\\ (\mathfrak{R}_1 q)_{i+1} - (\mathfrak{R}_1 q)_i\end{pmatrix},$$

3. Solve ODE's system $\rightarrow \delta^{n+1},\ \dot{\delta}^{n+1},\ \underline{\zeta}^{n+1},\ \dot{\underline{\zeta}}^{n+1}$,
4. Ghost cell $\rightarrow u_0^{n+1} = \left(\underline{\zeta}^{n+1}, -\frac{R}{2}\dot{\delta}^{n+1}\right)$.

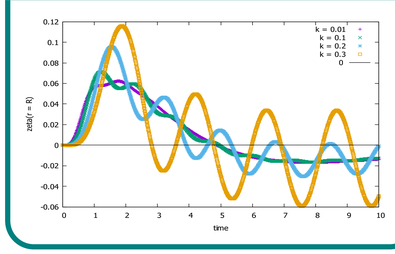
## Parameters

$\varepsilon = 0.3$, $\kappa \in \{0.01, 0.1, 0.2, 0.3\}$, $\delta_{in} = 0.1$, $(\zeta_{in}, q_{in}) = 0$, $T_{final} = 10$, $R = 4$, 200 cells.

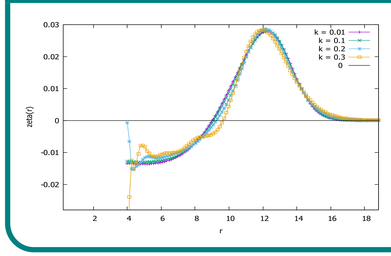
### $\delta(t)$



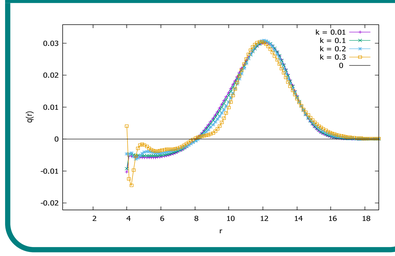

### $\underline{\zeta}(t)$




### $\zeta(x, T_{final})$




### $q(x, T_{final})$




## References

[1] G.Beck, E.Contentin, L.Martaud. Freely floating cylinder on a 3d fluid governed by the boussinesq equations in the axisymmetric without swirl case. In preparation. 2025.

[2] G.Beck, D.Lannes, L.Weynans. A numerical method for wave-structure interactions in the Boussinesq regime. Preprint. 2024.

[3] E.Bocchi. Floating structures in shallow water: local well-posedness in the axisymmetric case. SIAM J. Math. Anal. 52 (1), 306-339 (2020). 2019.