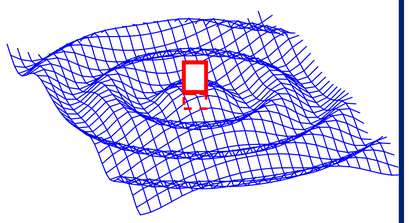
Vertical motion of a freely floating cylinder in Boussinesq regime.

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3D situation



Hypothesis

- Axisymetry without swirl,
  - Discharge: $q(t,r)\vec{e_r}$,
  - Wave's elevation: $\zeta(t,r)$,
- The fluid fit the bottom of the solid in $\mathcal{I}$,
- Regime:

$$\varepsilon \in [0,1), \quad \kappa \in (0,1).$$

Augmented formulation

$$\begin{cases} \partial_t \zeta + \mathrm{d}_r q = 0, \\ \partial_t q + \partial_r \mathfrak{R}_1 \left(\zeta + \varepsilon \frac{\zeta^2}{2} + \varepsilon \frac{q^2}{h} \right) + \varepsilon \mathfrak{R}_0 \frac{q^2}{rh} = \partial_t \underline{q} G, \end{cases} \qquad r > R \qquad (3)$$

$$\underline{q} = \lim_{\eta \to 0} q_e(R+\eta) = -\frac{R}{2}\dot{\delta},$$

and

$$\begin{pmatrix} \text{modified Newton} \\ \text{hidden trace eq} \end{pmatrix} \quad \begin{pmatrix} \tau_\kappa^2(\varepsilon\delta)\ddot{\delta} \\ \kappa^2 \underline{\ddot{\zeta}} \end{pmatrix} = \mathfrak{F}\left(\delta, \dot{\delta}, \underline{\zeta}, \underline{\dot{\zeta}}, \underline{\mathfrak{R}_1 f}\right), \qquad (4)$$

with $\mathfrak{R}_\alpha$, $\alpha \in \{0,1\}$ dispersive nonlocal operators defined as

$$\begin{cases} \left[1 - \kappa^2 \left(\partial_r + \alpha \frac{\cdot}{r}\right)\left(\partial_r + (1-\alpha)\frac{\cdot}{r}\right)\right] \mathfrak{R}_\alpha u = u, \quad \forall r > R, \\ (1-\alpha)\mathfrak{R}_\alpha u_{|_{r=R}} + \alpha \partial_r \mathfrak{R}_\alpha u_{|_{r=R}} = 0. \end{cases}$$

and G dispersive boundary layer defined as $\begin{cases} D_\kappa G = 0, \\ G(r=R) = 1. \end{cases}$

How to get the modified Newton's equation

Combining (2) with (1) in $\mathcal{I}$ leads to obtain the modified Newton's equation with

$$\tau_\kappa^2(\varepsilon\delta) := \tau_{buoy}^2 + \kappa^2 + \frac{R^2}{8(1+\varepsilon\delta)} > \tau_{\text{buoy}}$$

How to get the hidden equation

$$D_\kappa \partial_t q + \varepsilon d_r \frac{q^2}{h} + h \partial_r \zeta = 0, \quad D_\kappa = (1 - \kappa \partial_r \mathrm{d}_r)$$

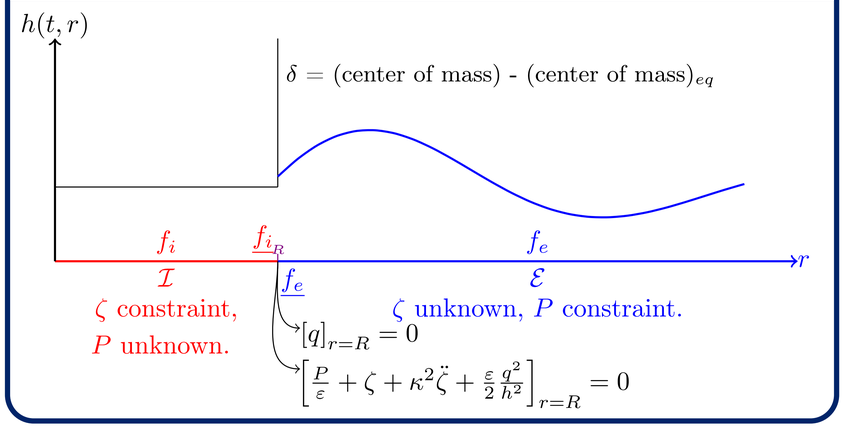
$$\Downarrow \mathfrak{R}_0$$

$$\partial_t q + \partial_r \mathfrak{R}_1 \left(\zeta + \varepsilon \frac{\zeta^2}{2} + \varepsilon \frac{q^2}{h} \right) + \varepsilon \mathfrak{R}_0 \frac{q^2}{rh} = \underline{\partial_t q} G$$

$$\Downarrow \kappa^2 \mathrm{d}_r$$

$$\kappa^2 \ddot{\zeta} + \zeta + \varepsilon \frac{\zeta^2}{2} + \varepsilon \frac{q^2}{h} = \ddot{\delta}\, \kappa^2 \frac{R}{2} d_r G + \mathfrak{R}_1 \left(\zeta + \varepsilon \frac{\zeta^2}{2} + \varepsilon \frac{q^2}{h} \right) + \varepsilon \kappa^2 \mathrm{d}_r \mathfrak{R}_0 \frac{q^2}{rh}, \quad r \geq R$$

1D Radial situation



Modelisation

- Boussinesq-Abbott

$$\begin{cases} \partial_t \zeta + \mathrm{d}_r q = 0, \\ D_\kappa \partial_t q + \varepsilon \mathrm{d}_r \frac{q^2}{h} + h \partial_r \zeta = -h \partial_r \frac{P}{\varepsilon}, \\ h = 1 + \varepsilon\zeta, \\ \mathrm{d}_r := \partial_r + \frac{\cdot}{r}, \quad D_\kappa = (1 - \kappa^2 \partial_r \mathrm{d}_r), \end{cases} \qquad (1)$$

- Newton's motion law

$$\tau_{buoy}^2 \ddot{\delta} + \frac{1}{\varepsilon} m = \frac{1}{\mathcal{I}} \int_{\mathcal{I}} \frac{P}{\varepsilon}. \qquad (2)$$

Local wellposedness

With $L_r^2 = L^2(rdr, (R, +\infty))$, we define the norms:

$$\| \cdot \|_{H_r^1}^2 = \| \cdot \|_{L_r^2}^2 + \| \mathrm{d}_r \cdot \|_{L_r^2}^2, \quad \| \cdot \|_{H_\kappa^2}^2 = \| \cdot \|_{H_r^1}^2 + \kappa^2 \| \partial_r \mathrm{d}_r \cdot \|_{L_r^2}.$$

The Augmented formula is locally wellposed in

$$(\zeta,\ q,\ \delta,\ \dot{\delta},\ \underline{\zeta},\ \underline{\dot{\zeta}}) \in \mathscr{C}^0\left([0,T), H_r^1 \times H_\kappa^2 \times \mathbb{R}^4\right),$$

$$T = \mathscr{O}\left(\min\left(\varepsilon^{-1}\kappa,\ \varepsilon^{-1/2}\right)\right)$$

Sketch of the proof: Fixed point argument based on the iterative scheme

$$\begin{cases} \partial_t \zeta^{k+1} + \mathrm{d}_r q^{k+1} = 0, \\ \partial_t q^{k+1} + \partial_r \mathfrak{R}_1 \left(\zeta^{k+1} + \varepsilon \frac{\zeta^{k^2}}{2} + \varepsilon \frac{q^{k^2}}{h^k} \right) + \varepsilon \mathfrak{R}_0 \frac{q^{k^2}}{rh^k} = \partial_t \underline{q}^{k+1} G, \end{cases} \qquad r > R$$

& Same for the linearized version of (4)

Return to the equilibirum in the linear case $\varepsilon = 0$

With initial conditions $(\zeta, q, \delta, \dot{\delta})_{t=0} = (0, 0, \delta_0 \neq 0, 0)$, δ reads in the Laplace domain $\widehat{\delta}(s) = \widehat{H}(s)\delta_0$,

$$\widehat{H}(s) := \frac{2\tau_\kappa^2(0)s + R\sqrt{1+\kappa^2 s^2}B(s)}{s\left(2\tau_\kappa^2(0)s + R\sqrt{1+\kappa^2 s^2}B(s)\right) + 2},$$

$$B(s) := \frac{K_0\left(\frac{Rs}{\sqrt{1+\kappa^2 s^2}}\right)}{K_1\left(\frac{Rs}{\sqrt{1+\kappa^2 s^2}}\right)}, \quad s \in \{\eta + i\omega, \eta > 0\} \cup \{i\omega, 0 < |\omega| < \kappa^{-1}\}.$$

Theorem on the decay of δ

$\delta \in \mathscr{C}^\infty(\mathbb{R}^+)$ and:

- $\delta \in H^2(\mathbb{R}^+)$ but $\forall \beta \in (0,2], \delta, \dot{\delta}, \ddot{\delta} \notin L^2(t^\beta dt, \mathbb{R}^+)$
- $\delta, \dot{\delta}$ and $\ddot{\delta}$ decay slower than $\mathcal{O}(t^{-1/2})$.

With viscosity, $\delta^{(k)}$ decays faster than $\mathcal{O}(t^{-1-k})$ and slower than $\mathcal{O}(t^{-3/2-k})$ for $k \in \{0,1\}$.

Numerical simulations

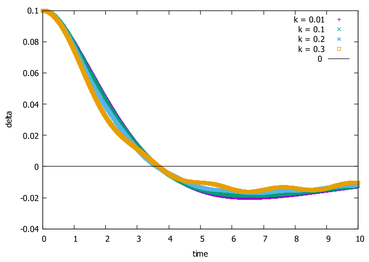
Finite volume method of [3] adapted to the 2D situation, if we know $(\zeta^n, q^n, \delta^n, \dot{\delta}^n, \underline{\zeta}^n, \underline{\dot{\zeta}}^n)$ at step n:

1. Solve the elliptic problems $\rightarrow \mathfrak{R}_1$ and $\mathfrak{R}_0$,
2. Solve (ζ^{n+1}, q^{n+1}) with (3),
3. Solve ODE's system (4) $\rightarrow \delta^{n+1}$, $\dot{\delta}^{n+1}$, $\underline{\zeta}^{n+1}$, $\underline{\dot{\zeta}}^{n+1}$,
4. Fill the ghost cell $\rightarrow (\underline{\zeta}^{n+1}, -\frac{R}{2}\dot{\delta}^{n+1})$.

$\delta(t)$

Settings of the simulation: $\varepsilon = 0.3$, $\delta_0 = 0.1$, $(\zeta, q)_{t=0} = 0$, $T_{final} = 10$, $R = 4$, 200 cells.

Presentation of the simulation of $\delta(t)$ for t in $[0, T_{final})$ in the case of the return to the equilibrium for different values of κ.



References

[1] G. Beck, E. Contentin, L. Martaud, Freely floating cylinder on a 3D fluid governed by the Boussinesq equations in the axisymmetry without swirl case. Preprint, 2025,

[2] G.Beck, E. Contentin, L. Martaud, The return to the equilibrium of a freely floating cylinder in the Boussinesq regime. WAVES 2026 - 17th International Conference on Mathematical and Numerical Aspects of Wave Propagation, June 2026, Montreal, Canada Preprint, 2026,

[3] G. Beck, D. Lannes, L. Weynans, A numerical method for wave-structure interactions in the Boussinesq regime. ESAIM: Mathematical Modelling and Numerical Analysis, 2025, 59 (6).